3.1 Introduction to Determinants

Recursive Definition of a Determinant

Recall

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

Example 1. Compute the determinant of the following matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det A = a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$= a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$$

where A_{11}, A_{12}, and A_{13} are obtained from A by deleting the first row and one of the columns.

For any $n \times n$ (square) matrix A, let A_{ij} denote the submatrix obtained by deleting the ith row and jth column of A. For example,

$$A = \begin{bmatrix} 1 & 0 & 5 & 6 \\ 2 & 4 & 0 & 8 \\ 6 & 5 & 7 & 4 \\ 1 & 3 & 5 & 4 \end{bmatrix}, \qquad A_{32} = \begin{bmatrix} 1 & 5 & 6 \\ 2 & 0 & 8 \\ 1 & 5 & 4 \end{bmatrix}$$

Definition (Determinant)

For $n \geq 2$, the determinant of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\begin{aligned} \det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j} \end{aligned}$$

Given $A = [a_{ij}]$, the (i, j)-cofactor of A is the number C_{ij} given by

$$\boxed{C_{ij} = (-1)^{i+j} \det A_{ij}} \quad (*)$$

Then

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

This formula is called a **cofactor expansion across the first row** of A.

The plus or minus sign in the (i, j)-cofactor depends on the position of a_{ij} in the matrix, regardless of the sign of a_{ij} itself. The factor $(-1)^{i+j}$ determines the following checkerboard pattern of signs:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}$$

Theorem 1. The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the i th row using the cofactors in $(*)$ is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

$C_{ij} = (-1)^{i+j} \det A_{ij}$

The cofactor expansion down the j th column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Example 2. $\begin{vmatrix} 1 & -2 & 4 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -4 & -3 & 5 \\ 2 & 0 & 3 & 5 \end{vmatrix}$ A

$|A| = \det A$

First we expand along the second row, then expand along either the third row or the second column of the remaining matrix

$$\det A = 3 \cdot (-1)^{2+3} \cdot \begin{vmatrix} 1 & -2 & 2 \\ 2 & -4 & 5 \\ 2 & 0 & 5 \end{vmatrix}$$

$$= (-3)\left(2 \cdot (-1)^{3+1} \cdot \begin{vmatrix} -2 & 2 \\ -4 & 5 \end{vmatrix} + 5 \cdot (-1)^{3+3} \cdot \begin{vmatrix} 1 & -2 \\ 2 & -4 \end{vmatrix} \right)$$

$$=(-3)\Big(2\cdot(-10+8)+5\cdot(0)\Big)=(-3)\cdot(-4)=12 .$$

or

$$=(-3)\left((-2)\cdot(-1)^{1+2}\cdot\begin{vmatrix}2&5\\2&5\end{vmatrix}+(-4)(-1)^{2+2}\cdot\begin{vmatrix}1&2\\2&5\end{vmatrix}\right)$$

$$=(-3)\big(0+(-4)\cdot 1\big)=12 .$$

Theorem 2. If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

Example 3. Compute the determinant.

$$\begin{vmatrix} 3 & 0 & 0 & 0 \\ 7 & -2 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 3 & -8 & 4 & -3 \end{vmatrix}$$

By Thm 2, $\det A = 3 \times (-2) \times 3 \times (-3) = 54$

Recall

upper triangular $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$.

lower triangular. $\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

Example 4. Explore the effect of an elementary row operation on the determinant of a matrix. State the row operation and describe how it affects the determinant.

(i) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \underline{ad - bc} \quad , \quad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = \underline{bc - ad} = -(ad - bc)$$

The row operation swaps rows 1 and 2 of the matrix, and the sign of the determinat is reversed.

(ii) $\begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 5+3k & 4+2k \end{bmatrix}$

$$\begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix} = 12 - 10 = 2$$

$$\begin{vmatrix} 3 & 2 \\ 5+3k & 4+2k \end{vmatrix} = 3\cdot(4+2k) - 2\cdot(5+3k) = 12 + \cancel{6k} - 10 - \cancel{6k} = 2$$

The row operation replaces row 2 by $k \times R1 + R2$, and the determinant is unchanged.

Exercise 5. Let $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$. Write $5A$. Is $\det 5A = 5 \det A$?

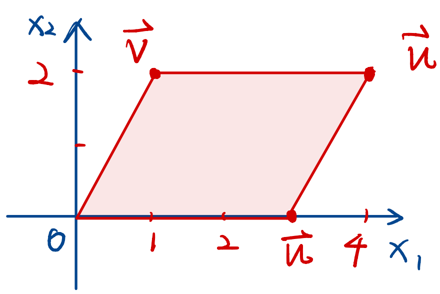
$$5A = \begin{bmatrix} 15 & 5 \\ 20 & 10 \end{bmatrix}$$

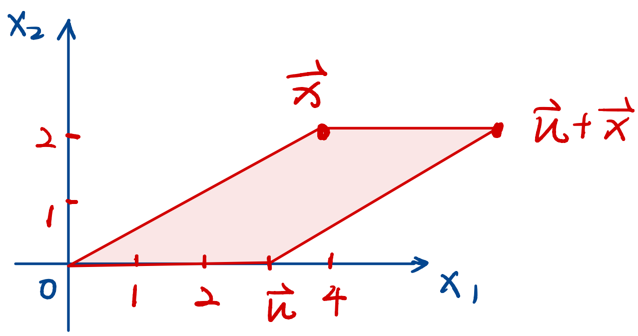
No. $\det 5A = 150 - 100 = 50$, $\det A = 6 - 4 = 2$

So $\det 5A = 5^2 \cdot \det A$

Exercise 6. Let $\mathbf{u} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Compute the area of the parallelogram determined by $\mathbf{u}, \mathbf{v}, \mathbf{u} + \mathbf{v}$, and $\mathbf{0}$, and compute the determinant of $[\mathbf{u} \quad \mathbf{v}]$. How do they compare? Replace the first entry of $\mathbf{v}$ by an arbitrary number x, and repeat the problem. Draw a picture and explain what you find.

ANS. The area of the parallelogram determined by $\mathbf{u} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{u} + \mathbf{v}$, and $\mathbf{0}$ is 6, since the base of the parallelogram has length 3 and the height of the parallelogram is 2 . By the same reasoning, the area of the parallelogram determined by $\mathbf{u} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x \\ 2 \end{bmatrix}, \mathbf{u} + \mathbf{x}$, and $\mathbf{0}$ is also 6.





Also, note that $\det [\mathbf{u} \quad \mathbf{v}] = \det \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} = 6$, and $\det [\mathbf{u} \quad \mathbf{x}] = \det \begin{bmatrix} 3 & x \\ 0 & 2 \end{bmatrix} = \mathbf{6}$. The determinant of the matrix whose columns are those vectors that define the sides of the parallelogram adjacent to $\mathbf{0}$ is equal to the area of the parallelogram